

# ANOTHER PROOF OF WRIGHT'S INEQUALITIES

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**ABSTRACT.** We present a short way of proving the inequalities obtained by Wright in [*Journal of Graph Theory*, 4: 393 – 407 (1980)] concerning the number of connected graphs with  $\ell$  edges more than vertices.

## 1. PRELIMINARIES

For  $n \geq 0$  and  $-1 \leq \ell \leq \binom{n}{2} - n$ , let  $c(n, n + \ell)$  be the number of connected graphs with  $n$  vertices and  $n + \ell$  edges. Quantifying  $c(n, n + \ell)$  represents one of the fundamental tasks in the theory of random graphs. It has been extensively studied since the Erdős-Rényi's paper [3]. The generating functions associated to the numbers  $c(n, n + \ell)$  are due to Sir E. M. Wright in a series of papers including [11, 12]. He also obtained the asymptotic formula for  $c(n, n + \ell)$  for every  $\ell = o(n^{1/3})$ . Using different methods, Bender, Canfield and McKay [1], Pittel and Wormald [8] and van der Hofstad and Spencer [9] were able to determine the asymptotic value of  $c(n, n + \ell)$  for all ranges of  $n$  and  $\ell$ .

For  $\ell \geq -1$ , let  $W_\ell$  be the exponential generating function (EGF, for short) of the family of connected graphs with  $n$  vertices and  $n + \ell$  edges. Thus,  $W_\ell(z) = \sum_{n=0}^{\infty} c(n, n + \ell) \frac{z^n}{n!}$ . Let  $T(z)$  be the EGF of the Cayley's rooted labeled trees. It is well known that  $T(z) = z e^{T(z)} = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!}$  (see for example [4, 5]). Among other results, Wright proved that the functions  $W_\ell(z)$ ,  $\ell \geq -1$ , can be expressed in terms of  $T(z)$ . Such results allowed penetrating and precise analysis when studying random graphs processes as it has been shown for example in the giant paper [5]. Throughout the rest of this note, all formal power series are univariate. Therefore, for sake of simplicity we will often omit the variable  $z$  so that  $T \equiv T(z)$ ,  $W_i \equiv W_i(z)$  and so on.

We need the following notations.

**Definition.** If  $A$  and  $B$  are two formal power series such that for all  $n \geq 0$  we have  $[z^n] A(z) \leq [z^n] B(z)$  then we denote this relation  $A \preceq B$  or  $A(z) \preceq B(z)$ .

The aim of this note is to provide an alternative and generating function based proof of the inequalities obtained by Sir Wright in [12] (in particular, he used numerous intermediate lemmas). More precisely, Wright obtained the following.

**Theorem (Wright 1980).** Let  $b_1 = \frac{5}{24}$  and  $c_1 = \frac{19}{24}$ . Define recursively  $b_\ell$  and  $c_\ell$  by

$$(1) \quad 2(\ell + 1)b_{\ell+1} = 3\ell(\ell + 1)b_\ell + 3 \sum_{t=1}^{\ell-1} t(\ell - t)b_t b_{\ell-t}, \quad (\ell \geq 1)$$

and

$$(2) \quad \begin{aligned} 2(3\ell + 2)c_{\ell+1} &= 8(\ell + 1)b_{\ell+1} + 3\ell b_\ell + (3\ell + 2)(3\ell - 1)c_\ell \\ &+ 6 \sum_{t=1}^{\ell-1} t(3\ell - 3t - 1)b_t c_{\ell-t}, \quad (\ell \geq 1) \end{aligned}$$

Then, for all  $\ell \geq 1$

$$(3) \quad \frac{b_\ell}{(1 - T(z))^{3\ell}} - \frac{c_\ell}{(1 - T(z))^{3\ell-1}} \preceq W_\ell(z) \preceq \frac{b_\ell}{(1 - T(z))^{3\ell}}.$$

(3) is known as *Wright's inequalities* and such results has been extremely useful in the enumerative study of graphs as well as in the theory of random graphs [2, 5, 6, 7, 10].

Our proof of (3) is based upon two ingredients:

**Fact 1.** We know that the EGFs  $W_\ell$  satisfy  $W_{-1} = T - \frac{T^2}{2}$ ,  $W_0 = -\frac{1}{2} \log(1 - T) - \frac{T}{2} - \frac{T^2}{4}$  and

$$(4) \quad (1 - T) \vartheta_z W_{\ell+1} + (\ell + 1) W_{\ell+1} = \left( \frac{\vartheta_z^2 - 3\vartheta_z}{2} - \ell \right) W_\ell + \frac{1}{2} \sum_{k=0}^{\ell} (\vartheta_z W_k) (\vartheta_z W_{\ell-k}), \quad (\ell \geq 0),$$

where  $T = T(z)$ ,  $W_k = W_k(z)$  and  $\vartheta_z = z \frac{\partial}{\partial z}$  corresponds to marking a vertex (such combinatorial operator consists to choose a vertex among the others). For the combinatorial sense of (4), we refer the reader to [1, 5] or [11].

**Fact 2.** Let  $A$  and  $B$  be two formal power series and  $\ell \in \mathbb{N}$ . If  $(1 - T) \vartheta_z A + (\ell + 1) A \preceq (1 - T) \vartheta_z B + (\ell + 1) B$  then  $A \preceq B$ .

To prove Fact 2, fix  $\ell \geq 0$ . We write

$$(5) \quad B(z) - A(z) = \sum_{n=0}^{\infty} (b_n - a_n) \frac{z^n}{n!} \quad \text{and} \quad \forall n, c_n = b_n - a_n.$$

Suppose that  $(1 - T) \vartheta_z A + (\ell + 1) A \preceq (1 - T) \vartheta_z B + (\ell + 1) B$ . We then have

$$(6) \quad \begin{aligned} n! [z^n] ((1 - T(z)) \vartheta_z (B(z) - A(z)) + (\ell + 1) (B(z) - A(z))) = \\ (n + \ell + 1)c_n - \sum_{k=1}^n \binom{n}{k} k^{k-1} (n - k) c_{n-k} \geq 0. \end{aligned}$$

It is now easily seen that  $\forall n, c_n \geq 0$ . Therefore,  $A \preceq B$ .

Our proof of (3) is divided into two parts each of each are given in the next Sections.

## 2. PROOF OF $W_\ell \preceq \frac{b_\ell}{(1-T)^{3\ell}}$

Define the family  $(\overline{W}_\ell)_{\ell \geq 0}$  as  $\overline{W}_0 = -\frac{1}{2} \log(1-T)$  and for  $\ell \in \mathbb{N}^*$ ,  $\overline{W}_\ell = \frac{b_\ell}{(1-T)^{3\ell}}$ . Observe that we have  $W_0 \preceq \overline{W}_0$  and  $W_1 \preceq \overline{W}_1$  has been proved in [12]. Now, we can proceed by induction. Suppose that for  $2 \leq i \leq \ell$ ,  $W_i \preceq \overline{W}_i = \frac{b_i}{(1-T)^{3i}}$  and let us prove that  $W_{\ell+1} \preceq \overline{W}_{\ell+1} = \frac{b_{\ell+1}}{(1-T)^{3\ell+3}}$ . Simple calculations show that

$$(7) \quad \left( \frac{\vartheta_z^2 - \vartheta_z}{2} \right) \overline{W}_\ell \preceq \frac{\vartheta_z^2}{2} \overline{W}_\ell \preceq \frac{3\ell(3\ell+2)}{2} \frac{b_\ell}{(1-T)^{3\ell+4}} - \frac{3\ell(3\ell+2)}{2} \frac{b_\ell}{(1-T)^{3\ell+3}},$$

$$(8) \quad (\vartheta_z \overline{W}_0) (\vartheta_z \overline{W}_\ell) \preceq \frac{3\ell b_\ell}{2} \frac{b_\ell}{(1-T)^{3\ell+4}} - \frac{3\ell b_\ell}{2} \frac{b_\ell}{(1-T)^{3\ell+3}} \quad \text{and}$$

$$(9) \quad \frac{1}{2} \sum_{p=1}^{\ell-1} (\vartheta_z \overline{W}_p) (\vartheta_z \overline{W}_{\ell-p}) \preceq \frac{1}{2} \left( \sum_{p=1}^{\ell-1} 9p(\ell-p)b_p b_{\ell-p} \right) \left( \frac{1}{(1-T)^{3\ell+4}} - \frac{1}{(1-T)^{3\ell+3}} \right).$$

Summing (7), (8), (9), using the recurrence (1) and the induction hypothesis, we find that

$$(10) \quad (1-T)\vartheta_z W_{\ell+1} + (\ell+1)W_{\ell+1} \preceq \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+3}}.$$

Since

$$(11) \quad (1-T)\vartheta_z \overline{W}_{\ell+1} + (\ell+1)\overline{W}_{\ell+1} = \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{2(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+3}}$$

by Fact 2, we have  $\overline{W}_{\ell+1} \succeq W_{\ell+1}$ .

## 3. PROOF OF $\frac{b_\ell}{(1-T)^{3\ell}} - \frac{c_\ell}{(1-T)^{3\ell-1}} \preceq W_\ell$

Define  $\underline{W}_0 = W_0$  and for  $\ell \in \mathbb{N}^*$ ,  $\underline{W}_\ell = \frac{b_\ell}{(1-T)^{3\ell}} - \frac{c_\ell}{(1-T)^{3\ell-1}}$ . As before, we shall proceed by induction. We have  $\underline{W}_0 \preceq W_0$  and

$$(12) \quad W_1 - \underline{W}_1 = \frac{13}{12(1-T)} - \frac{1}{2} - \frac{T}{8} + \frac{T^2}{24} \succeq \frac{13}{12} \left( \frac{1}{(1-T)} - T - 1 \right) = \frac{13T^2}{12(1-T)} \succeq 0.$$

Suppose that for  $2 \leq k \leq \ell$ ,  $\underline{W}_k = \frac{b_k}{(1-T)^{3k}} - \frac{c_k}{(1-T)^{3k-1}} \preceq W_k$ . We have to prove that  $\underline{W}_{\ell+1} = \frac{b_{\ell+1}}{(1-T)^{3\ell+3}} - \frac{c_{\ell+1}}{(1-T)^{3\ell+2}} \preceq W_{\ell+1}$ . For this purpose, define  $\Psi_{\ell+1}$  as

$$(13) \quad \begin{aligned} \Psi_{\ell+1} = & \left( \frac{\vartheta_z^2 - 3\vartheta_z}{2} - \ell \right) \underline{W}_\ell + (\vartheta_z \underline{W}_0) (\vartheta_z \underline{W}_\ell) + \frac{1}{2} \sum_{k=1}^{\ell-1} \left( \vartheta_z \underline{W}_k - \frac{(3\ell-1)c_\ell}{(1-T)^{3\ell}} \right) (\vartheta_z \underline{W}_{\ell-k}) \\ & - \left( \frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}} \right), \end{aligned}$$

where  $\alpha_\ell$ ,  $\beta_\ell$ ,  $\gamma_\ell$  and  $\delta_\ell$  are given by

$$(14) \quad \begin{aligned} \alpha_\ell &= \frac{(7\ell+4)c_{\ell+1}}{2} - 3(\ell+1)b_{\ell+1} - \frac{3}{4}\ell b_\ell + \frac{(3\ell-1)(3\ell+4)}{4}c_\ell \\ &+ \frac{1}{2} \sum_{t=1}^{\ell-1} (3t-1)c_t(3\ell-3t-1)c_{\ell-t}, \end{aligned}$$

$$(15) \quad \begin{aligned} \beta_\ell &= -\frac{(3\ell+2)c_{\ell+1}}{2} + 2(\ell+1)b_{\ell+1} - \frac{3}{4}\ell b_\ell - \frac{(3\ell-1)(3\ell+4)}{4}c_\ell \\ &- \frac{1}{2} \sum_{t=1}^{\ell-1} (3t-1)c_t(3\ell-3t-1)c_{\ell-t}, \end{aligned}$$

$$(16) \quad \gamma_\ell = \frac{\ell b_\ell}{2} + \frac{(3\ell-1)c_\ell}{2} \quad \text{and} \quad \delta_\ell = -\frac{\ell-1}{2}c_\ell.$$

Rewriting the formal power series  $\frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}}$  as follows

$$(17) \quad \begin{aligned} &\frac{(7\ell+4)/2 c_{\ell+1} - 3(\ell+1)b_{\ell+1} - 3/4\ell b_\ell}{(1-T)^{3\ell+2}} - \frac{(3\ell+2)/2 c_{\ell+1} - 2(\ell+1)b_{\ell+1} + 3/4\ell b_\ell}{(1-T)^{3\ell+1}} \\ &+ (3\ell-1)(3\ell+4)c_\ell \left( \frac{1}{(1-T)^{3\ell+2}} - \frac{1}{(1-T)^{3\ell+1}} \right) \\ &+ \frac{2\ell b_\ell}{2(1-T)^{3\ell}} + \left( \frac{(3\ell-1)c_\ell}{2(1-T)^{3\ell}} - \frac{(\ell-1)c_\ell}{2(1-T)^{3\ell-1}} \right), \end{aligned}$$

it is easily seen that if the quantity (coming from the denominators of the 2 first terms of the above equation)

$$(18) \quad (2\ell+1)c_{\ell+1} - (\ell+1)b_{\ell+1} - \frac{3}{2}\ell b_\ell \geq 0$$

then  $\frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}} \succeq 0$ . (We used  $1/(1-T)^a \succeq 1/(1-T)^b$  if  $a \geq b$ ).

Using (1) and (2), after simple algebra we have (18). Therefore by construction, RHS of (4)  $\succeq \Psi_{\ell+1}$ . After nice cancellations, it yields

$$(19) \quad \Psi_{\ell+1} = \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{2(\ell+1)b_{\ell+1} + (3\ell+2)c_{\ell+1}}{(1-T)^{3\ell+3}} + \frac{(2\ell+1)c_{\ell+1}}{(1-T)^{3\ell+2}}.$$

Remarking that  $(1-T)\vartheta_z \underline{W}_{\ell+1} + (\ell+1) \underline{W}_{\ell+1} = \Psi_{\ell+1}$ , we have completed the proof of  $\underline{W}_{\ell+1} \preceq W_{\ell+1}$ .

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